Quantum cavity modes in spatially extended Josephson systems

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We report a theoretical study of the macroscopic quantum dynamics in spatially extended Josephson systems. We focus on a Josephson tunnel junction of finite length placed in an externally applied magnetic field. In such a system, electromagnetic waves in the junction are excited in the form of cavity modes manifested by Fiske resonances, which are easily observed experimentally. We show that in the quantum regime various characteristics of the junction as its critical current I_c, width of the critical current distribution σ, escape rate Γ from the superconducting state to a resistive one, and the time-dependent probability $P(t)$ of the escape are influenced by the number of photons excited in the junction cavity. Therefore, these characteristics can be used as a tool to measure the quantum states of photons in the junction, e.g. quantum fluctuations, coherent and squeezed states, entangled Fock states, etc.

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Great interest is currently attracted to experimental and theoretical studies of macroscopic quantum phenomena in diverse Josephson systems [1, 2, 3]. Most of such systems contain just one or few *lumped* Josephson junctions. At low temperatures, quantum-mechanical effects such as macroscopic quantum tunneling, energy level quantization, and coherent oscillations of the Josephson phase have been observed [2, 3]. The important method allowing to study the macroscopic energy levels in Josephson coupled systems is the microwave spectroscopy [2, 3]. As the frequency ω of an externally applied microwave radiation matches the energy level separation, one can observe resonant absorption and Rabi oscillations due to the population of excited levels.

An interaction between Josephson systems and *quantized electromagnetic fields* opens new frontiers in research. Similar type of interaction arises between an atom and electromagnetic cavity modes in quantum electrodynamics (QED). Recently, such QED-inspired experiments have been performed with superconducting charge qubits [4] and flux qubits [5, 6] coupled to on-chip resonators. In these experiments, the quantum degree of freedom of a lumped Josephson system couples to the quantized electromagnetic field of a superconducting cavity located on the same chip, as illustrated in Fig. 1(a). The excitation of *cavity modes* CMs in an external cavity leads to an appearance of an oscillating current flowing through the Josephson circuit. Such an oscillating current excites transitions between macroscopic energy levels of the Josephson phase. Quantum regime of a weak resonant interaction between the Josephson phase φ and CMs is described by a bilinear Hamiltonian $\hat{H} \propto \varphi(\hat{a}+\hat{a}^{+})$, where $\hat{a}^{+}(\hat{a})$ are the operators of creation (annihilation) of a particular CM. This Hamiltonian corresponds to the famous Jaynes-Cummings model [7, 8] and therefore, a quantum regime of a lumped Josephson circuit incorporated in an external transmission line can be mapped to a problem of a single atom weakly interacting with CMs. In this regime, many fascinated phenomena as mixture of different Rabi frequencies, creation of entangled states of CMs, a single atom maser behavior, can be observed [9]. The interaction via CMs can be used to couple superconducting qubits [10, 11, 12].

As we turn to *spatially extended* Josephson systems, e.g. long Josephson junctions, Josephson junction parallel arrays and ladders, the spatially-dependent Josephson phase $\varphi(x)$ can also display macroscopic quantum effects as tunneling and energy level quantization. For a Josephson vortex trapped in a long Josephson junction, these phenomena have been studied theoretically [13] and observed in experiments [14]. An additional, interesting property of spatially extended junctions is that they can support the propagation of electromagnetic waves forming CMs *inside* the junction itself, see Fig. 1(b). When the Josephson system is biased in the resistive state the resonant interaction of such CMs with ac Josephson current leads to the classical resonances in the current-voltage characteristics known as Fiske steps [15, 16]. The quantum regime of such resonances at finite voltages in systems containing few small Josephson junctions has been theoretically considered in Ref. [17]. The next step naturally appearing in a study of macroscopic quantum phenomena in spatially extended Josephson systems is the quantum regime of interaction between the Josephson phase and *intrinsic CMs* of the system.

In a current-biased Josephson junction, the spatially-averaged Josephson phase difference φ oscillates at Josephson plasma frequency $\omega_p = \omega_{p0}\left[1-(I/I_{c0})^2\right]^{1/4}$ depending on the dc bias I. Here, ω_{p0} and I_{c0} are the plasma frequency at zero bias and the nominal value of the critical current of the junction, respectively. If the Josephson plasma frequency ω_p matches the frequency of CMs, the interaction between the Josephson phase and CMs becomes resonant.

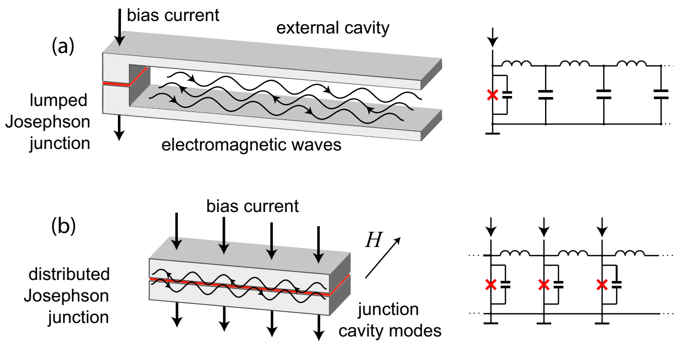

FIG. 1: (a) Schematic view of lumped Josephson junction embedded in an external cavity and its equivalent scheme. (b) Distributed Josephson junction subject to an external magnetic field with excited internal (Fiske) CMs and its equivalent scheme.

In this paper, we consider the alternative, non-resonant interaction between the Josephson phase φ and intrinsic CMs of the junction. Such an interaction can be realized in long Josephson junctions or their discrete versions – parallel Josephson arrays, often also called as Josephson transmission lines. We show that the switching from the superconducting state to a resistive one, i.e. the Josephson phase escape can be facilitated or suppressed by excitation of cavity modes.

Moreover, we argue that measurements of the switching current I_c, the width of its distribution σ, and the time-dependent probability $P(t)$ of the escape allow to obtain information on the *quantum state* of intrinsic CMs in the junction.

In order to quantitatively analyze the macroscopic quantum phenomena appearing due to the interaction of plasma oscillations with CMs, we consider a Josephson junction of a finite length L. The Josephson junction is characterized by the time and coordinate dependent Josephson phase $\varphi(x,t)$. In the presence of an externally applied magnetic field the Josephson phase can be written as

$$\varphi(x,t) \;=\; \frac{2\pi\Phi x}{\Phi_0 L} + \tilde{\varphi}(x,t) \quad , \tag{1}$$

where Φ is the external magnetic flux, and $\Phi_0 = hc/2e$ is the flux quantum [18]. The system is described by Hamiltonian:

$$H = E_{J0}\int_0^L \left[\frac{1}{2\omega_{p0}^2}\tilde{\varphi}_t^2 + \frac{\lambda_J^2}{2}\tilde{\varphi}_x^2 + U(x,\tilde{\varphi})\right]\frac{dx}{L} \ ,$$

$$U(x,\tilde{\varphi}) \;=\; E_{J0}\left[1-\cos\left(\frac{2\pi\Phi x}{\Phi_0 L}+\tilde{\varphi}\right) - i\tilde{\varphi}\right] \ , \tag{2}$$

where λ_J and E_{J0} are the Josephson penetration length and the Josephson energy, accordingly. The normalized dc bias $i = I/I_{c0}$ can be changed to tune the effective potential $U(x,\tilde{\varphi})$ of the junction.

Next, we represent the time-dependent Josephson phase by a sum of electromagnetic CMs and a "center mass" Josephson phase $\chi(t)$, i.e.

$$\tilde{\varphi}(x,t) \;=\; \chi(t) + \sum_n Q_n(t)\cos(k_n x), \tag{3}$$

where the wave numbers of cavity modes $k_n = \pi n/L$, with $n = 1, 2....$ Substituting expression (3) in Eq. (2) and assuming that the amplitudes of cavity modes $Q_n(t)$ are small (so that we can neglect anharmonic interaction between CMs [19]), we obtain the Hamiltonian in the following form:

$$H = \frac{E_{J0}}{2\omega_{p0}^2}\left[\dot{\chi}^2 + \frac{1}{2}\sum_n \dot{Q}_n^2 + \frac{\omega_n^2}{2}Q_n^2\right] + U(\chi, Q_n),$$

$$U(\chi, Q_n) \;=\; E_{J0}\sum_{n,m} a_{nm}Q_nQ_m - E_J(H)\cos\chi - i\chi \ , \tag{4}$$

where $\omega_n = \omega_{p0}\lambda_J k_n$. The magnetic field dependent coupling energy is $E_J(H) = E_{J0}\frac{I_c(H)}{I_{c0}}$, where $I_c(H)$ is the magnetic field suppressed nominal value of the critical current, i.e.

$$I_c(H) = I_{c0}\left|\frac{\sin(\frac{\pi\Phi}{\Phi_0})}{\frac{\Phi}{\Phi_0}}\right| \quad .$$

The χ-dependent coefficients

$$a_{nm} \;=\; \frac{1}{2}\int \frac{dx}{L}\cos(k_n x)\cos(k_m x)\cos\left(\frac{2\pi\Phi x}{\Phi_0 L} + \chi\right)$$

determine the strength of interaction between the CMs and the center mass Josephson phase.

As $E_J(H) \gg \hbar\omega_{p0}$, the switching from the superconducting state to a resistive one occurs at the dc bias close to its critical value, i.e. $\delta = [I_c(H) - I]/I_c(H) \ll 1$, and the quantum-mechanical Hamiltonian (4) can be written in the following form:

$$\hat{H} \;=\; \hat{H}_0 + \sum_n \hat{H}_n + \hat{H}_{int} \quad , \tag{5}$$

where the first term

$$H_0 \;=\; \frac{\omega_{p0}^2}{2E_{J0}}\hat{P}_\chi^2 + E_J(H)\left[\delta\chi - \frac{\chi^3}{6}\right] \tag{6}$$

describes the dynamics of homogeneous (plasma) oscillations of the Josephson phase with momentum operator $\hat{P}_\chi$. The second term is the Hamiltonian of noninteracting CMs

$$H_n \;=\; \frac{\omega_{p0}^2}{E_{J0}}\hat{P}_{Q_n}^2 + E_{J0}\frac{\omega_n^2}{4\omega_{p0}^2}Q_n^2 \ , \tag{7}$$

where $\hat{P}_Q = -i\hbar\frac{\partial}{\partial Q}$ is the operator of generalized momentum . Here, we have neglected a small renormalization of CMs spectrum due to the presence of plasma oscillations of the Josephson phase χ. The last term in (5) describes an interaction between the Josephson phase and CMs:

$$\hat{H}_{int} \;=\; -\,E_J\chi\sum_n a_n Q_n^2\;, \tag{8}$$

where $a_n = \frac{1}{2}\int\frac{dx}{L}\cos^2(k_n x)\cos(\frac{2\pi\Phi x}{\Phi_0 L})$. In the absence of an external magnetic field all coefficients $a_n = 1/4$. However, in the presence of a magnetic field characterizing by magnetic flux, $\Phi \simeq m\Phi_0$, the coefficient $a_m = -1/8$, and other coefficients are small.

If the size of the Josephson system is not very large ($L \le \lambda_J$), all frequencies $\omega_n \gg \omega_{p0}$ and, therefore, the interaction between the CMs and the plasma oscillations is non-resonant. Thus we can use a time-averaged expression for the interaction Hamiltonian, $\hat{H}_{int} = E_J\chi\sum_n a_n < Q_n^2 >$, where $< ... >$ is the time average. The mean switching current $\bar{I}_c$ is written as

$$\bar{I}_c = I_c(H) - I_{c0}\sum_n a_n < Q_n^2 > \quad . \tag{9}$$

The excitation of CMs results in a nonzero value of $< Q_n^2 >$, which leads to a *suppression* of the average switching current in the absence of magnetic field. However, when the magnetic field is applied, the fluctuation-free critical current $I_c(H)$ is strongly suppressed and excitation of CMs results in an unusual *increase* of the mean switching current (a_n values become negative in this case).

Switching from the superconducting (zero-voltage) state to the resistive state occurs at random values of the external current I. This process is characterized by the probability $P(I)$. If we neglect the CMs, the Josephson phase escape in the quantum regime is determined by macroscopic tunneling, and the width σ of the distribution $P(I)$ strongly depends on magnetic field:

$$\sigma \;\simeq\; \left(\frac{\hbar\omega_{p0}}{E_{J0}}\right)^{4/5} I_{c0}^{2/5} I_c(H)^{3/5} \quad . \tag{10}$$

Quantum fluctuations induced by zero-point oscillations of CMs lead to a *saturation* of σ as a function of magnetic field at the level

$$\sigma_{CM} \;\simeq\; I_{c0}\frac{\hbar\omega_{p0}L}{E_{J0}\lambda_J}\;. \tag{11}$$

The typical dependence of the width of the critical current distribution on magnetic flux applied to the junction is shown by solid line in Fig. 2.

In the following, we present a quantitative analysis of the influence of quantum-mechanical properties of CMs on the process of escape from the superconducting state

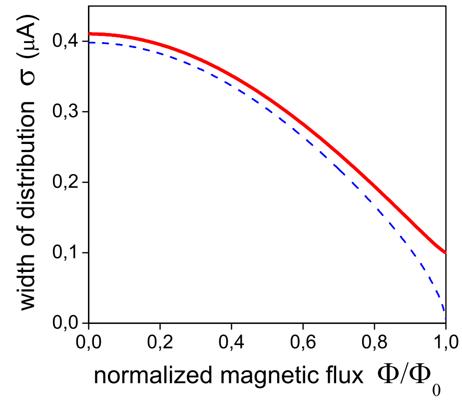

FIG. 2: The width of the switching current probability distribution as a function of magnetic field with (solid line) and without (dashed line) taking into account zero-point oscillations of CMs. Here we have taken typical parameters for the junction as $I_{c0} = 100\mu A$, and $\hbar\omega_{p0}/E_{J0} = 10^{-3}$, the junction size $L = \lambda_J$.

to the resistive state of the junction. In the quantum regime the process of escape is determined by the tunneling of phase χ and the switching rate is given by [1, 2, 3]

$$\Gamma(I,Q_n) \propto e^{-\frac{48(\sqrt{2}E_J(H)E_{J0})^{1/2}}{5\hbar\omega_{p0}}\left(\delta-\frac{I_{c0}}{I_c(H)}\sum_n a_n Q_n^2\right)^{5/4}}\;. \tag{12}$$

In experiment, the measured characteristics is usually the time-dependent probability $P(I,t)$ of finding the junction in the zero-voltage state. Since, Q_n are the random variables, this probability can be written as

$$P(I,t) = \int \rho(Q_n)\exp\left[-t\,\Gamma(I,Q_n)\right]dQ_n\;. \tag{13}$$

Here, $\rho(Q_n)$ is the quantum-mechanical probability distribution of CMs in the junction.

If we neglect the excitation of CMs, $P_0(t) \simeq \exp(-t/\tau_0)$ displays a standard exponential decay with time. Here, $\tau_0 = 1/\Gamma(I,0)$ is the dc bias dependent lifetime of the zero-voltage state, see Eq. (12). This behavior is shown in Fig. 3 by the dashed line. The deviation of $P(t)$ from the exponential dependence allows to characterize the quantum-mechanical properties of CMs. The situation becomes especially interesting for *nonequilibrium and nonclassical states* of the cavity modes. These are the zero-point oscillations, the chaotic, coherent and squeezed states, and various entangled states well known in quantum optics [7, 8].

As a particular example, first, we consider zero-point oscillations in CMs. In this case the $\rho(Q_n)$ is given as

$$\rho_{\rm zp}(Q_n) = \prod_n\left(\frac{E_{J0}\omega_n}{2\pi\hbar\omega_{p0}^2}\right)^{1/2}\exp\left(-\frac{E_{J0}}{2\omega_{p0}^2\hbar}\sum_n\omega_n Q_n^2\right)\;. \tag{14}$$

Substituting (14) in Eq. (13) and calculating all integrals

over Q_n, we obtain for $\Phi \approx \Phi_0$

$$P_{CM}^{\rm zp}(t) \simeq \left(\frac{\gamma_1 \tau_0}{t}\right)^{\gamma_1}, \quad t \gg \gamma_1 \tau_0, \qquad (15)$$

where the magnetic field dependent parameter $\gamma_1 = \frac{\pi \lambda_J}{3L}\left(\frac{I_c(H)}{I_{c0}}\right)^{1/2}(2\delta)^{-1/4}$. For small times $t < \gamma_1 \tau_0$ the influence of the equilibrium CMs quantum noise is small, and the dependence $P(t) \simeq \exp(-t/\tau_0)$ restores. Notice here, that for small external magnetic field and not very large number of excited CMs the exponential dependence of $P(I,t)$ is always valid.

Various nonequilibrium quantum-mechanical states of CMs can be induced by applying external microwaves. The interesting case is the *coherent state* of a single CM excited at a frequency $\omega = \omega_1$. In this case all values of Q_n except for Q_1 are small, and $\rho(Q_1)$ is given by

$$\rho_{coh}(Q_1) \simeq \exp\left[-\frac{E_J \omega_1}{2\omega_{p0}^2 \hbar}(Q_1 - \eta)^2\right] \quad , \qquad (16)$$

where $\eta \propto \sqrt{W}$ is determined by the power W of external microwave radiation. For a Josephson junction subject to an external magnetic field, we obtain that the excitation of the coherent state of a single CM leads to the dependence $P(t)$ in the form

$$P_{CM}^{\rm coh}(t) \simeq \exp\left\{-\gamma_2 \ln^2 \frac{t}{\gamma_2 \tau_{\rm coh}}\right\}, \quad t \gg \gamma_2 \tau_{\rm coh}. \qquad (17)$$

The parameter $\gamma_2 \simeq \frac{\lambda_J}{L}\frac{\hbar \omega_{p0} I_c(H)}{E_{J0} I_{c0} \eta}\delta^{-1/2}$ depends on both magnetic field and the power of microwave radiation. Here $\tau_{\rm coh} = 1/\Gamma(I, Q_1 = \eta)$, see Eq. (12). Similarly to the case of a quantum noise in CMs, at short times $t < \gamma_2 \tau_{\rm coh}$ the $P(I,t)$ decays exponentially with time, i.e. $P_0(t) \simeq \exp(-t/\tau_{\rm coh})$.

Another interesting quantum-mechanical state of CMs is the chaotic state of a single CM induced by microwave radiation [7, 8]. In this case the power W of external microwave radiation determines only the mean photon number $\bar{m}$, i.e. $\bar{m} \propto W$. As the mean photon number $\bar{m}$ is relatively large the probability distribution $\rho(Q_n)$ takes the form

$$\rho_{\rm ch}(Q_n) = \left(\frac{E_{J0}\omega_1}{2\pi \omega_{p0}^2 \hbar}\bar{m}\right)^{1/2} \exp\left(-\frac{E_{J0}\omega_1}{2\hbar \omega_{p0}^2 \bar{m}} Q_1^2\right) \qquad (18)$$

Therefore, a strongly excited chaotic state of a single CM should show the same time-dependent probability $P(t)$ as in the zero-point oscillations case (15), for which γ_1 is replaced by the microwave power dependent parameter $\gamma_3 = \gamma_1/\bar{m}$.

The probability $P(t)$ of finding the junction in the zero-voltage state is shown in Fig. 3 for various states of CMs. We note that more complicated states as squeezed states or Fock states (e.g., N photons in one mode $n = 1$) can

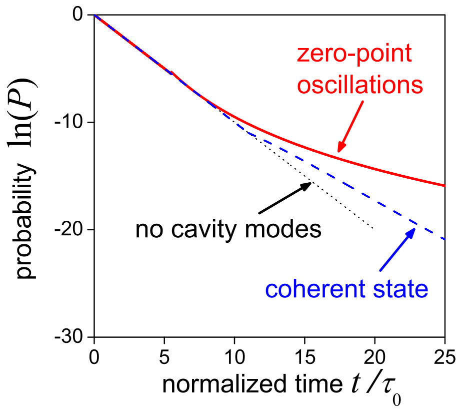

FIG. 3: Time-dependent probability $P(t)$ of finding the junction in the zero-voltage state: without taking into account CMs (dotted line) and with taking into account quantum fluctuations induced by CMs (solid line) and coherent state of CMs (dashed line). The parameters are chosen as $\delta = 5\ 10^{-3}$, $I_c(H)/I_c(0) = 0.2$, $\hbar\omega_{p0}/E_{J0} = 10^{-4}$, $\eta = 1.5\ 10^{-4}$ and the junction size $L = 0.27\lambda_J$. For simplicity, we have taken $\tau_0 = \tau_{\rm coh}$.

be prepared by using pulsed technique and intrinsic nonlinearity of cavity modes [20]. The entanglement of Fock states will manifest itself by oscillations of $P(t)$ dependence. For realistic values of junction parameters we obtain $\gamma_1 \simeq \gamma_2 = 7$, and the deviations of $P(t)$ from the exponential decay should be detectable experimentally.

In conclusion, we have shown that the excitation of cavity modes in distributed Josephson junction or parallel arrays of junctions manifests itself by either enhancement or suppression of the escape rate from the superconducting state, depending on applied magnetic field. This effect is due to a renormalization of the potential barrier for the escape which, in turn, depends on the quantum state of the cavity mode. The important characteristics of the cavity mode quantum electrodynamics, namely the probability distribution of the CMs $\rho(Q)$ can be detected experimentally by measuring the temporal decay of the switching probability $P(t)$ given by Eqs. (12) and (13).

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[18] Strictly speaking the Eq. (1) is valid in the limit of $L \ll \lambda_J$, where λ_J is the Josephson penetration length. However, our analysis is qualitatively correct also for junctions of a size $L \simeq \lambda_J$ and subjected to large magnetic field when the critical current is strongly suppressed.

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